LU-Factorization Method to Solve Fully Fuzzy Linear Programming Problem as Multi Objective Linear Programming Problem

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Abstract- This paper presents the LU Factorization Method to solve fully fuzzy linear programming problem with the help of multi objective linear programming problem when the constraint matrix and the cost coefficients are fuzzy in nature. LU method is based on the fact that a square matrix can be factorized into the product of unit lower triangular matrix (L) and upper triangular matrix (U). This method gives direct solution without iteration and it is also explained with an illustrative example.

Key words: Fully Fuzzy linear programming problem, Trapezoidal fuzzy number, Triangular fuzzy number, Multi Objective Fuzzy Linear Programming Problem, LU Factorization Method, Unit Lower Triangular Matrix, Upper Triangular Matrix.

1. INTRODUCTION

Linear programming is one of the most important operational research techniques. All linear programming problems are concerned with maximization (or minimization) of a linear function subject to a set of linear constraints. The simplex method of linear programming was developed by Prof. G. B. Dantzig in 1947. Linear programming has been applied to solve many real world problems but it fails to deal with imprecise data. So many researchers succeed in capturing vague and imprecise information by fuzzy linear programming problem (FLPP) (Bellman and Zadeh, (1970), Bezdek, (1993), Zhang, (2003)). The first formulation of fuzzy linear programming (FLP) was proposed by Zimmermann (1978). An application of fuzzy optimization techniques to linear programming problems with multiple objectives has been presented by Zimmermann(1978). Tanaka et. Al(1974) presented a fuzzy approach to multi objective linear programming problems. Negoita has formulated FLPP with fuzzy coefficient matrix, Zhang G. et al. (2003) formulated a FLPP as four objective constrained optimization problem where the cost coefficients are fuzzy and also presented it's solution. Thakre P.A. et al(2009) proposed the method to the solution of fuzzy linear programming problem.

In this paper fully fuzzy linear programming problem with the help of multi objective linear programming problem is solved using the LU Factorization Method. This method gives direct solution without iteration and it is explained with an illustrative example.

2. PRELIMINARIES

Definition 1:
A subset A of a set X is said to be fuzzy set if $\mu_A: X \to [0,1]$, where μ_A denote the degree of belongingness of A in X.

Definition 2 :
A Fuzzy set A of a set X is said to be normal if $\mu_A(x) = 1$, $\forall x \in X$.

Definition 3:
The height of A is defined and denoted as $h(A) = \sup_{x \in X} \mu_A(x)$.

Definition 4:
The α- cut and strong α- cut is defined and denoted respectively as,
$\alpha_A = \{ x / \mu_A(x) \geq \alpha \}$
$\alpha^+_A = \{x/\mu_A(x) > \alpha\}$.

Definition 5 :
Let $\tilde{a}, \tilde{b}$ be two fuzzy numbers , their sum is defined and denoted as,

$$\mu_{\tilde{a}+\tilde{b}}(z) = \sup \min_{z=u+v}\{\mu_{\tilde{a}}(u), \mu_{\tilde{a}}(v)\}$$

When $0 \leq \lambda \in R$.

Definition 6:
If a fuzzy number $\tilde{a}$ is fuzzy set A on R , it must possess at least following three properties:

(1) $\mu_{\tilde{a}}(x) = 1$
(2) $\{x \in R / \mu_{\tilde{a}}(x) > \alpha\}$ is a closed interval for every $\alpha \in (0,1]$.
(3) $\{x \in R / \mu_{\tilde{a}}(x) > 0\}$ is bounded and it is denoted by $[a_\lambda^L, a_\lambda^R]$.

Theorem 1: A fuzzy set A on R is convex if and only if $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min[\mu_A(x_1), \mu_A(x_2)]$, for all $x_1, x_2 \in$ X and for all $\lambda \in [0,1]$ where min denotes the minimum operator.

Theorem 2: Let $\tilde{a}$ be a fuzzy set on R , Then$\tilde{a} \in f$ (R) if and only if $\mu_{\tilde{a}}$ satisfies

$$\mu_{\tilde{a}}(x) = \begin{cases} 1, & for\ x \in [m,n] \\ L(x), & for\ x < m \\ R(x), & for\ x > n. \end{cases}$$

Where $L(x)$ is the right continuous monotone increasing function , $0 \le L(x) \le 1$, and $\lim_{x\to\infty} L(x) = 0$, $R(x)$ is a left continuous monotone decreasing function , $0 \le R(x) \le 1$, and $\lim_{x\to\infty} R(x) = 0$.

3. FULLY FUZZY LINEAR PROGRAMMING PROBLEM

Fully Fuzzy linear programming problem (FFLPP) with cost of decision variables and coefficient matrix of constraints in fuzzy nature is defined (Thakre P.A. et al(2009))as

$\langle \tilde{c}, x\rangle = f_i(x_j) = f_i(x) = \text{Max } \tilde{Z} = \sum_{j=1}^{n} \tilde{c}_j x_j$

Subject to

$\sum_{j=1}^{n} \tilde{A}_{ij}\ x_j \le \tilde{B}_i$; $1 \le i \le m$, there exist $x_j > 0$ ……………..(1)

$$\mu_{a_{ij}}(x) = \begin{cases} 1, & for\ x < a_{ij} \\ \frac{(a_{ij}+d_{ij}-x)}{d_{ij}} & for\ a_{ij} \le x \le a_{ij}+d_{ij} \\ 0, & for\ x \ge a_{ij}+d_{ij} \end{cases}$$

$$\mu_{B_i}(x) = \begin{cases} 1, & for\ x \le a_{ij} \\ \frac{(b_i+p_i-x)}{p_i}, & for\ b_i \le x \le b_i+p_i \\ 0, & for\ b_i+p_i \le x \end{cases}$$

Consider triangular fuzzy number A which can be represented by three crisp numbers *s*,*l*,*r*.

(1) $\Rightarrow \langle \tilde{c}, x\rangle = f_i(x_j) = \text{Max } \sum_{j=1}^{n} \tilde{c}_j x_j$

Such that

$\sum_{x\ge 0}(s_{ij}, l_{ij}, r_{ij})\, x_{ij} \le (t_i, u_i, v_i)$,

$0 \le i \le m$, $0 \le j \le n$

where, $A_{ij} = \langle s_{ij}, l_{ij}, r_{ij}\rangle$, $B_{ij} = \langle t_i, u_i, v_i\rangle$ are fuzzy numbers.

Theorem 3: For any two triangular fuzzy numbers $A = \langle s_1, l_1, r_1\rangle$ and $B = \langle s_2, l_2, r_2\rangle$,A≤B if and only if $s_1 \le s_2$, $s_1 - l_1 \le s_2 - l_2$ and $s_1 + r_1 \le s_2 + r_2$

Above problem can be rewritten as

$\langle \tilde{c}, x\rangle = f_i(x_j) = \text{Max} \sum_{j=1}^{n} \tilde{c}_j x_j$

Such that
$$\sum_{j=1}^{n} s_{ij} x_j \le t_i$$
$$\sum_{j=1}^{n}(s_{ij} - l_{ij})\, x_j \le t_i - u_i$$
$$\sum_{j=1}^{n}(s_{ij} + r_{ij})x_j \le t_i + v_i, x_i \ge 0 \quad \text{……………(2)}$$

where the membership functions of $\tilde{c}_j(x)$ is

$$\mu_{\tilde{c}_j}(x) = \begin{cases} 0, & x \le \alpha_j \\ \frac{x-\alpha_j}{\beta_j-\alpha_j}, & \alpha_j \le x \le \beta_j \\ 1, & \beta_j \le x \le \gamma_j \\ \frac{\eta_i - x}{\eta_i - \gamma_j}, & \gamma_j \le x \le \eta_j \\ 0, & \eta_j < x \end{cases}$$

Definition 7:

A point x^* $\in$ X is said to be an optimal solution to the FLPP if,

$\langle \tilde{c}, x^*\rangle \ge \langle \tilde{c}, x\rangle$ for all $x \in$ X.

4. SOLVING FFLPP AS MULTI OBJECTIVE LPP USING LU-FACTORIZATION METHOD

For the above (FFLPP) ,the multi objective linear programming problem with fuzzy coefficients can be formulated as

$$\max_{x\in X}\{f_1(x), f_2(x), \dots\dots . f_k(x)\}$$

Subject to (2)

where f_i: $R^n \rightarrow R^i$

Where R be the set of all real numbers and R^n be an n-dimensional Euclidean space .

By considering the weighting factor, the MOLPP is defined as

$$\text{Max}_{x\in X}\{w_1 f_1(x), w_2 f_2(x), \dots, w_k f_k(x)\}$$

i.e, $\text{Max}_{x\in X}\ w = \sum_{m=1}^{k} w_m f_m(x)$

subject to (2)

We write this L. P. P. (Chinchole and Bhadane (2014))in the following form

To find: *w*

Subject to:

$$-w_1 f_1(x) - w_2 f_2(x) - w_3 f_3(x) - \dots - w_k f_k(x) + w \le 0,$$
$$\sum_{j=1}^{n} s_{ij} x_j \le t_i$$
$$\sum_{j=1}^{n}(s_{ij} - l_{ij})\, x_j \le t_i - u_i$$
$$\sum_{j=1}^{n}(s_{ij} + r_{ij})x_j \le t_i + v_i, x_i \ge 0$$
$$-x_1, -x_2, -x_3, \dots, -x_{n-1}, -w \le 0.$$

Now we can consider the system of linear equations **AX = B**

LU Factorization technique is used to solve the problem and solution for different weights are found.

Solving a system of linear equations by using a factorization technique for matrices is called LU Factorization. This factorization involves two matrices, one unit lower triangular matrix and one upper triangular matrix.

Steps to solve a system of linear equations using an LU decomposition:

1. Set up the system of linear equations in *n* variables, $x_1, x_2, \dots\dots, x_n$ as a matrix equation where AX=B, where A=[a_{ij}] is an *n*x *n* matrix of real coefficients, X=[x_i] is nx1matrix of variables and B=[b_i] is nx1matrix of constants.

2. Find the unit lower triangular matrix and the upper triangular matrix U such that (LU)=A. This will yield the equation (LU)X=B.
3. Let Y=UX. Then solve the equation LY=B for Y.
4. Take the values for Y and solve the equation Y=UX for X. This will give the solution to the system AX=B.

In this way, the objective function is considered as a constraint and w is considered as a variable.

Case I

If the number of inequalities is equal to the number of variables:
Then LU Factorization method is applied to the system of linear equations AX=B . First, we get the matrix Y as an initial iteration and then the matrix X as final iteration.

Case II

If the number of inequalities is less than the number of variables:
Add the inequalities in the system till the number of inequalities equals the number of variables. We can add the inequalities in the system as below:
Consider the first constraint in given L. P. P.:
$a_{11}x_1+a_{12}x_2+a_{13}x_3+\ldots+a_{1,m}x_m\leq b_2$,
Choose the non zero coefficient in this inequality $a_{2j}\neq 0$and add the inequality $a_{21}x_j\leq b_2$ in the system.
Continuing in this way till number of inequalities reaches to the number of variables.

Case III

If the number of variables is less than the number of inequalities:
If there are less number of variables than that of inequalities, then we introduce that much number of slack variables in the appropriate inequalities and add **+1**on R.H. S. of each of that inequalities.

Case IV

If there is a zero row in upper triangular matrix U , then the given problem has an infeasible solution and we can stop the process.

5. NUMERICAL EXAMPLE

Consider the following FFLPP
Max $f_i(x_1,x_2) = (7,10,14,25)\, x_1+(20,25,35,40)\, x_2$
Subject to the constraints
$(3,2,1)\, x_1+ (6,4,1)\, x_2 \leq (13,5,2)$
$(4,1,2)\, x_1+ (6,5,4)\, x_2\leq (7,4,2)$
The above FFLPP can be written using (2) as
Max $f(x_1,x_2) = (7,10,14,25)\, x_1+(20,25,35,40)\, x_2$
Subject to the constraints
$3x_1+6x_2\leq 13$
$4x_1+6x_2\leq 7$
$x_1+2x_2\leq 8$
$3x_1+ x_2 \leq 3$
$4x_1+7x_2 \leq 15$
$6x_1+10x_2\leq 9$
$x_1 , x_2 \geq 0$
MOLPP: Max
$(7x_1+ 20x_2 , 10x_1+25x_2 , 14x_1+35x_2 , 25x_1+40x_2)$
Subject to
$3x_1+6x_2\leq 13$
$4x_1+6x_2\leq 7$
$x_1+2x_2\leq 8$
$3x_1+ x_2 \leq 3$
$4x_1+7x_2 \leq 15$
$6x_1+10x_2\leq 9$
$x_1 , x_2 \geq 0$
MOLPP
$\text{Max}(w)=(w_1(7x_1+20x_2)+w_2(10x_1+25x_2)$
$+w_3(14x_1+35x_2)+w_4(25x_1+40x_2))$
Subject to
$3x_1+6x_2\leq 13$
$4x_1+6x_2\leq 7$
$x_1+2x_2\leq 8$
$3x_1+ x_2 \leq 3$
$4x_1+7x_2 \leq 15$
$6x_1+10x_2\leq 9$
$x_1 , x_2 \geq 0$
LU decomposition technique is used to solve the above MOLPP & the solutions are found for different weights.
For example , $w_1= 0 =w_4$, $w_2= 1 =w_3$
MOLPP
$\text{Max}(w)\ f(x_1,x_2) = 24x_1+60x_2$
Subject to
$3x_1+6x_2\leq 13$
$4x_1+6x_2\leq 7$
$x_1+2x_2\leq 8$
$3x_1+ x_2 \leq 3$
$4x_1+7x_2 \leq 15$
$6x_1+10x_2\leq 9$
$x_1 , x_2 \geq 0$
This L. P. P. can be written in the following form
To find: w
Subject to:
$-24x_1-60x_2+w \leq 0$
$3x_1+6x_2 +x_3\leq 13$
$4x_1+6x_2+x_4\leq 7$
$x_1+2x_2+x_5\leq 8$
$3x_1+ x_2+x_6 \leq 3$
$4x_1+7x_2 \leq 15$
$6x_1+10x_2\leq 9$
$-x_1 ,- x_2, -w \leq 0$
Solution by MATLAB
A = [-24 -60 1 0 0 0 0 3 6 0 1 0 0 0 4 6 0 0 1 0 0 1 2 0 0 0 1 0 3 1 0 0 0 0 1 4 7 0 0 0 0 0 6 10 0 0 0 0 0];
B = [0;14;8;9;4;15;9]
[L,U] = lu (A)
Y = L\b
X = U\y
Solution:
Y =
0

4.0000
5.9231
9.3750
2.0833
7.4583
8.9583

X =

-43.5000
27.0000
576.0000
-17.5000
20.0000
-1.5000
107.5000

Following table lists the solution for the above LPP for various weights and it also shows that the solutions are independent of weights(w_i,i=1,2,3,4) .

Sl.no	W_1	W_2	W_3	W_4	(x_1^*,x_2^*)
1	0	1	1	0	(-43.5,27)
2	0	1	0.5	0	(-43.5,27)
3	0.2	0.4	0.5	0.2	(-43.5,27)
4	0.1	0.2	0.3	0.4	(-43.5,27)
5	0	0.3	0	0.4	(-43.5,27)
6	0.2	0.4	0.6	0.8	(-43.5,27)
7	0.5	0	0.5	0	(-43.5,27)
8	0	1	1	0	(-43.5,27)
9	0	0	0	0.5	(-43.5,27)
10	0.3	0.1	1	1	(-43.5,27)
11	0.5	0.5	0.5	0.5	(-43.5,27)
12	0	0	0.5	0.5	(-43.5,27)
13	0.2	0.5	0.5	0.5	(-43.5,27)
14	0.1	0.2	0.3	0.4	(-43.5,27)
15	0	0.2	0	0.2	(-43.5,27)

The optimum solution is,
X_1=-43.5;X_2=27;Max w = 576.

6. CONCLUSION

Here, we discussed the solution of fully fuzzy linear programming problem with the help of multi objective constrained linear programming problem where constraint matrix and the cost coefficients are fuzzy quantities using LU -factorization method and also proved that the solutions are independent of weights.

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